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Evolution of the precessional motion of unbalanced gyrostats of variable structure ☆

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ABSTRACT

The precessional motion of an unbalanced gyrostat of variable structure when acted upon by dissipative and accelerating external and internal moments, which depend on the angular velocities of the bodies (the carrier and the rotor) is considered. A qualitative method of analysing the phase space of non-autonomous dynamical systems is developed, based on the determination of the curvature of the phase trajectory. The motion is analysed and the conditions for obtaining the required modes of nutational-precessional motion of unbalanced gyrostats of variable structure are synthesized using this method. A number of cases of the motion of a gyrostat of variable structure, including free motion, motion when there are constant internal and reactive moments and, also, under the action of the moments of resistance forces, proportional to the angular velocities, is investigated. The possible evolutions in the above-mentioned cases of motion and the causes of these evolutions are determined. The conditions for evolution with a decreasing amplitude of the nutational oscillations are obtained.

The basic cases of the motion of rigid bodies of variable structure have been described earlier in Refs 1, 2; the equations of motion were constructed on the basis of the "close action" hypothesis[1] and the "solidifying" principle[2]. The equations of motion of a gyrostat (coaxial bodies) of variable structure were described using the close action hypothesis and approximate solutions for the parameters of motion were obtained for linear laws of variation of the inertia-mass parameters.[3,4] A development of these results, associated with the investigation of the motion of unbalanced gyrostats of variable structure with non-linear laws of variation of the inertia-mass parameters under the action of internal and external perturbations, is given below.

1. Perturbed motion of an unbalanced gyrostat of variable structure

Consider the motion of a gyrostat of variable structure (mass) acted upon by dissipative and accelerating external moments which depend on the components of the angular velocities. Suppose a gyrostat consists of a dynamically symmetric body-carrier of constant structure and a rotor of variable structure, which remains dynamically symmetric when the mass changes. The fixed point O (Fig. 1) coincides with the initial geometrical position of the centre of mass of the system. The points O_n and O_r correspond to the centres of mass of the body-carrier and the body-rotor. We shall use the following systems of coordinates: $OXYZ$ is a fixed system, $Oxyz$ is the system associated with the body-carrier and $Ox_ry_rz_r$ is the system associated with the body-rotor. The rotor rotates along the longitudinal axis Oz_r which coincides with the axes of dynamic symmetry of the bodies. The imbalance of the gyrostat implies an inconstancy of the relative angular velocity of rotation of the rotor with respect to the body-carrier due to the action of an internal interaction moment M_r between the bodies. Suppose there is a reactive moment M_z^R about the longitudinal axis Oz_r.

We write the equations of motion of the system[3]

$$\tilde{A}(t)\dot{p} + (C(t) - A(t))qr + C_r(t)q\sigma = M_x^e(p, q, r, \sigma) \tag{1.1}$$

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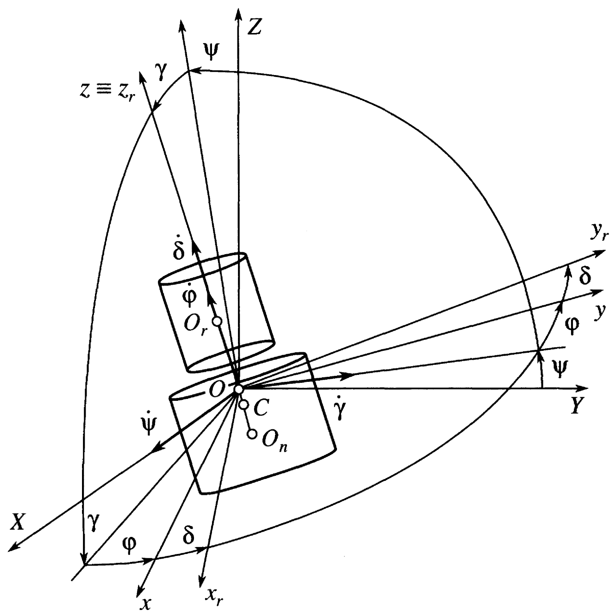

Fig. 1.

$$\tilde{A}(t)\dot{q} - (C(t) - A(t))pr - C_r(t)p\sigma = M_y^e(p, q, r, \sigma)$$

$$C(t)\dot{r} + C_r(t)\dot{\sigma} = M_z^R + M_z^e(r, \sigma), \quad C_r(t)(\dot{r} + \dot{\sigma}) = M_r + M_z^R + M_{z,r}^e(r, \sigma)$$

Here,

$$\tilde{A}(t) = A(t) - m(t)\rho^2(t), \quad A(t) = A_n + A_r(t), \quad C(t) = C_n + C_r(t)$$

A_n, C_n and $A_r(t), C_r(t)$ are the equatorial and longitudinal moments of inertia of the body-carrier and the rotor (with subscripts n and r respectively), $M_x^e(p, q, r, \sigma) = M_{x,n}^e + M_{x,r}^e$, $M_y^e(p, q, r, \sigma) = M_{y,n}^e + M_{y,r}^e$, $M_z^e(r, \sigma) = M_{z,n}^e + M_{z,r}^e$ are the moments of the external forces, $m(t)\rho^2(t)$ is a term which arises as a consequence of the geometrical displacement of the centre of mass relative to a fixed point[1,3] related to the change in the inertia-mass structure of the system, $\rho(t)$ is the varying distance from the fixed point to the centre of mass of the system along the longitudinal Oz axis, $m(t) = m_n + m_r(t)$ is the varying mass of the gyrostat, p, q and r are the projections of the angular velocity of the body-carrier onto the axes of the system $Oxyz$ and σ is the relative angular velocity of the rotor. A geometrical displacement of the centre of mass implies a change in the position of the centre of mass with reference to the system, rather than the motion of a point in a mechanical sense.

It is necessary to make some remarks regarding the inertia-mass quantities occurring in dynamic equations (1.1). The equatorial moment of inertia of the system $A(t)$ is calculated with respect to the fixed point O and the mass parameters of the rotor change with time. As a consequence of this, the position of the centre of mass of the system C is geometrically displaced with respect to the point O. On account of the change in the mass of the points within the rotor, the position of its centre of mass may also change (in this case, the point O_r is shifted along the Oz_r axis, while retaining the dynamic symmetry of the rotor). Suppose, for example, the mass of the rotor decreases. Then, at the same time, the centre of mass of system C is displaced along the Oz_r axis towards the centre of mass of the body-carrier O_n (Fig. 1). The quantity $\rho(t)$, that is equal to the distance from the fixed point O to the centre of mass of system C along the Oz axis, is calculated using the definition of the centre of mass and has the form

$$\rho(t) = \frac{l_n m_n + l_r(t) m_r(t)}{m_n + m_r(t)} \tag{1.2}$$

where $l_n = OO_n$ and $l_r(t) = OO_r$ are the distances from the centres of mass of the body-carrier and the body-rotor to the point O. Since, when $t = 0$, the centre of mass C and the point O coincide, we have $\rho(0) = 0$ ($l_r(0)m_r(0) = -l_n m_n$, $l_n < 0$, $l_r(0) > 0$).

We shall assume that the intrinsic central equatorial moments of inertia of the bodies $\bar{A}_n > 0$ and $\bar{A}_r(t) > 0$, calculated with respect to the centres of mass O_n and O_r, are known. Then, using the Huygens–Steiner theorem, the central equatorial moment of inertia of the system about the actual position of the centre of mass of C can be calculated:

$$\bar{A}(t) = [\bar{A}_n + m_n(|l_n| - |\rho(t)|)^2] + [\bar{A}_r(t) + m_r(t)(|l_r(t)| + |\rho(t)|)^2] > 0$$

Using the Huygens–Steiner theorem again, we link the central moment of inertia of the system about the point C with the moment of inertia of the system about the point O

$$A(t) = \bar{A}(t) + m(t)\rho^2(t) \tag{1.3}$$

It follows from the last relation, that $\tilde{A}(t) = \bar{A}(t) > 0$ in Eq. (1.1).

It is important to note that a finite time interval $t \in [0, T]$ is considered in the problem in which all the inertia-mass parameters of the system remain strictly positive, that is, the mechanical formulation of the problem is correct and the dynamical system (1.1) does not become singular.

The kinematic equations for the Krylov–Euler angles (Fig. 1)

$$\dot{\gamma} = p\sin\varphi + q\cos\varphi, \quad \dot{\psi} = \frac{1}{\cos\gamma}(p\cos\varphi - q\sin\varphi)$$
$$\dot{\varphi} = r - \frac{\sin\gamma}{\cos\gamma}(p\cos\varphi - q\sin\varphi), \quad \dot{\delta} = \sigma \tag{1.4}$$

where $\delta = \angle(Ox, Ox_r)$ is the angle of relative rotation of the rotor, have to be added to the dynamic equations.

We now introduce the following new variables, corresponding to the magnitude of the transverse angular velocity vector G and the angle F between this vector and the Oy axis:

$$p(t) = G(t)\sin F(t), \quad q = G(t)\cos F(t) \tag{1.5}$$

In the new variables, Eq. (1.1) are written as follows:

$$\dot{G} = \frac{1}{\tilde{A}(t)} f_G(G, F), \quad \dot{F} = -\frac{1}{\tilde{A}(t)}[C(t)r + C_r(t)\sigma - A(t)r + f_F(G, F)]$$
$$\dot{r} = \frac{M^e_{z,n} - M_r}{C_n}, \quad \dot{\sigma} = \frac{C(t)M_r}{C_r(t)C_n} + \frac{M^R_z + M^e_{z,r}}{C_r(t)} - \frac{M^e_{z,n}}{C_n} \tag{1.6}$$

The perturbing functions, characterizing the external actions, have the form

$$f_G(G, F) = (M^e_x\sin F + M^e_y\cos F), \quad f_F(G, F) = \frac{1}{G}(M^e_x\cos F - M^e_y\sin F)$$

We will now consider the case when the modulus of the transverse angular velocity of the body-carrier is small compared with the relative longitudinal velocity of rotation of the rotor:

$$\varepsilon = \sqrt{p^2 + q^2}/|\sigma| \ll 1 \tag{1.7}$$

and we will assume that the angles γ and ψ are quantities of the order of ε. Then, the angle of nutation θ (the angle between the OZ and Oz_i axes) is determined by the following approximate formula

$$\theta^2 \cong \gamma^2 + \psi^2 \tag{1.8}$$

Taking account of the assumptions which have been made and relations (1.5) and (1.6), the kinematic Eq. (1.4), after discarding terms starting from the second order of smallness, can be written in the form

$$\dot{\gamma} \cong G\cos\Phi(t), \quad \dot{\psi} \cong G\sin\Phi(t), \quad \dot{\varphi} \cong r, \quad \dot{\delta} = \sigma \tag{1.9}$$

where the expression

$$\Phi(t) = F(t) - \varphi(t) \tag{1.10}$$

determines the phase of the spatial oscillations.

2. A method of calculating the curvature for analysing the phase space of a non-autonomous dynamical system

We will now consider the motion of a phase point over the phase plane of any non-autonomous dynamical system, corresponding to two non-autonomous first order differential equations such as, for example, the first two equations of system (1.9) (Fig. 2) with respect to the phase plane $\{\gamma, \psi\}$. In the above-mentioned plane, a point will have components of the velocity $V_\gamma = \dot{\gamma}$, $V_\psi = \dot{\psi}$ and of the acceleration $W_\gamma = \ddot{\gamma}$, $W_\psi = \ddot{\psi}$ (1.9), and the curvature of its trajectory k is therefore calculated using relations (1.9) as follows:

$$k^2 = \frac{(\ddot{\gamma}\dot{\psi} - \ddot{\psi}\dot{\gamma})^2}{(\dot{\gamma}^2 + \dot{\psi}^2)^3} = \frac{\dot{\Phi}^2}{G^2} \tag{2.1}$$

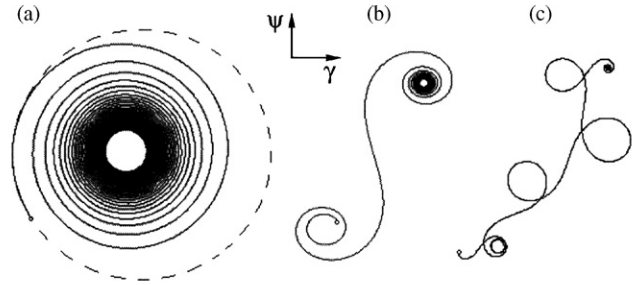

Fig. 2.

If the magnitude of the curvature increases monotonically, motion occurs along a curling spiral trajectory, similar to the trajectory in the neighbourhood of the stable focus (Fig. 2, *a*) but, if it decreases, it moves along an unwinding trajectory. By virtue of equalities (2.1), the condition for spiral curling can be written, when account is taken of the positiveness of G, in the form

$$k\dot{k} > 0 \Rightarrow \dot{\Phi}\ddot{\Phi}G - \dot{G}\dot{\Phi}^2 = P(t) > 0 \tag{2.2}$$

According to relation (1.8), the length of the radius vector of a phase point in the space $\{\gamma, \psi\}$ corresponds to the magnitude of the angle of nutation, and it is therefore necessary for condition (2.2) to be satisfied in the case of modes of motion with decreasing amplitude of the nutational oscillations.

We will first consider the motion of an autonomous system corresponding to a free gyrostat of constant structure. It follows from relations (2.1), (1.6) and (1.9) and the third equation of (1.1) that k = const (since $M_z^R = 0$ in this case):

$$k = -\frac{Cr_0 + C_r\sigma_0}{AG} \tag{2.3}$$

where $A = A_n + A_r$ = const and $C = C_n + C_r$ = const. Consequently, the phase trajectory (PT) will have the form of a circle of radius $R = 1|k|$ passing through the point $\{\gamma_0, \psi_0\}$ with centre on the perpendicular to the initial phase velocity vector $\mathbf{V}_0 = G_0[\sin\Phi(0), \cos\Phi(0)^T]$. Note that this circle corresponds to the motion with constant magnitude of the angle of nutation in the fixed system of coordinates, the OZ axis of which is orientated along the fixed direction of the kinetic momentum vector (at the same time, the origin of the system of phase coordinates $\{\gamma, \psi\}$ shifts to the centre of the circle).

We next consider a non-autonomous system which describes the motion of a gyrostat of variable structure. To analyse the possibility of satisfying condition (2.2), it is necessary to study the arrangement of the roots of the function $P(t)$, which describes the evolution of the form of the PT.

Different qualitative cases of the behaviour of the PT are possible depending on the number and arrangement of the roots of the function $P(t)$ (Fig. 2): 1) the function $P(t)$ is positive and has no roots in the interval $t \in [0, T]$; the PT spirally curls in this case (Fig. 2, *a*), 2) there is a single root and just one change in the monotonicity of the curvature of the PT (Fig. 2, *b*), 3) there are several roots and an alternation of unwinding and curling segments of the PT, and points of self-intersection also occur (Fig. 2, *c*).

We will now consider the problem when the variable moments of inertia and the moment of the reactive forces are known functions of time and the moment of the internal interaction between the carrier and the rotor is described by a known dependence on time or on the longitudinal angular velocities r and σ. If, at the same time, system (1.6) admits of a solution for the angular velocity parameters $r(t)$, $\sigma(t)$, $G(t)$, $F(t)$ in quadratures, then the function $P(t)$ can be written in explicit form. If expansion of the resulting quadratures in power series is used, it is possible to represent the function $P(t)$ in the form of a polynomial $P_N(t)$ of finite power N, which holds with a certain accuracy within its convergence interval.

The real, non-multiple roots of this polynomial determine the inflection points of the PT while the intervals in which the sign is maintained determine the time intervals of the monotonicity of its behaviour. By choosing the laws for the change in the inertia-mass parameters of the gyrostat and the initial conditions of motion, it is possible to change the coefficients of the polynomial $P_N(t)$, thereby affecting its roots and, consequently, the shape of the PT. Moreover, it is possible to ensure stability of the polynomial $P_N(t)$ (in the sense of the arrangement of the roots), which will be a sufficient condition for the monotonic behaviour of the PT. Standard techniques such as the use of the Routh–Hurwitz inequalities, Mikhailov's amplitude-phase criterion, etc[5] are entirely applicable for this purpose.

The motion of a gyrostat of variable structure can therefore be investigated by analysing the arrangement of the roots of the function $P(t)$ (or of the polynomial which approximates it) without subsequent solution of the Darboux problem associated with the integration of kinematic Eq. (1.9). We will next consider examples of the use of the method that has been developed to analyse the phase space of a gyrostat of variable structure.

3. Free motion of a balanced gyrostat of variable structure

We will consider the free motion of a balanced gyrostat (when there are no moments M_r, and M_z^R) of variable structure with moments of inertia which decrease linearly with time. We will determine the conditions under which motion with a monotonically decreasing amplitude of nutation occurs, which is a necessary condition in certain practical problems of the mechanics of space flight.[2,4] The linear laws for the change in the moments of inertia of the rotor correspond to a situation in which the mass of the rotor decreases linearly with time and uniformly throughout the whole volume:

$$m_r(t) = m_r - \nu t \; (\nu > 0,\, m_r = m_r(0),\, m_r(t) > 0,\, t \in [0, T])$$
$$A_r(t) = \alpha m_r(t), \quad C_r(t) = \beta m_r(t) \tag{3.1}$$

where α and β are constant quantities. Thus, for example, in the case of a continuous cylindrical shape of the rotor

$$\alpha = H^2/12 + R^2/4 + l_r^2, \quad \beta = R^2/2$$

where H is the height of the cylinder and R is its radius. We shall assume that there is no initial longitudinal rotation of the body-carrier, that is, $r_0 = 0$.

In this case, the equations of motion (1.6) take the form

$$G = G_0, \quad \dot{F} = -\frac{1}{\tilde{A}(t)}(C_r - ct)\sigma_0, \quad r = r_0 = 0, \quad \sigma = \sigma_0$$
$$\tilde{A}(t) = A - at - m(t)\rho^2(t) \tag{3.2}$$

where $A = A_n + \alpha m_r$ is the initial transverse moment of inertia of the gyrostat, $C_r = \beta m_r$ is the initial longitudinal moment of inertia of the rotor, and $a = \alpha\nu > 0$ and $c = \beta\nu > 0$ are the rates of decrease of the moments of inertia. Using relations (3.2), we have

$$P(t) = \frac{(C_r - ct)\sigma_0^2}{\tilde{A}^3(t)}\left[aC_r - c(A - m(t)\rho^2(t)) + (C_r - ct)\frac{d}{dt}(m(t)\rho^2(t))\right] \tag{3.3}$$

The quantity $\tilde{A}(t)$ is positive since it is equal to the central moment of inertia, and the positiveness of the expression in square brackets, which corresponds to the inequality

$$g(t) + a/A > c/C_r \tag{3.4}$$

where

$$g(t) = \frac{1}{AC_r}\left[cm(t)\rho^2(t) + (C_r - ct)\frac{d}{dt}(m(t)\rho^2(t))\right] \tag{3.5}$$

is therefore necessary and sufficient for condition (2.2) to be satisfied.

The condition for a decrease in the amplitude of nutation (3.4) generalizes the analogous condition[4] obtained earlier by another method

$$a/A > c/C_r \tag{3.6}$$

in which the quantity $g(t)$ was not taken into account. Since, in the case considered, there is a uniform change of mass in the volume of the rotor, its centre of mass maintains its position with respect to the system ($l_r = \text{const}$). The time dependence (1.2) of the distance from the fixed point O to the centre of mass of system C will have the form

$$\rho(t) = -\nu l_r t/(m - \nu t); \quad m = m_n + m_r(0) \tag{3.7}$$

In this case, the function (3.5) and its derivative are written as follows:

$$g(t) = \nu^2 l_r^2 \frac{2C_r mt - (\nu C_r + cm)t^2}{AC_r(m - \nu t)^2}, \quad \frac{dg}{dt} = \frac{2\nu^2 l_r^2 (C_r - ct)m^2}{AC_r(m - \nu t)^3}$$

It can be seen that the function $g(t)$ increases monotonically on the interval $t \in [0, C_r/c] \supset [0, T]$. At the initial instant of time, conditions (3.4) and (3.6) are identical but, as time passes, condition (3.4) intensifies compared with (3.6).

4. The motion of an unbalanced gyrostat of variable structure under the action of constant internal and reactive moments

We will now consider the motion of an unbalanced gyrostat under the action of constant longitudinal internal and reactive moments. As in Section 3, we shall assume that there is a uniform reduction in the mass of the rotor throughout its volume and that the linear laws (3.1) hold.

With these assumptions, Eq. (1.6) are written in the form

$$\dot{G} = 0, \quad \dot{F} = -\frac{(C_n + C_r - ct - A + at)r + (C_r - ct)\sigma}{A - at - m(t)\rho^2(t)}$$
$$\dot{r} = -\frac{M_r}{C_n}, \quad \dot{\sigma} = \frac{(C_n + C_r - ct)M_r}{(C_r - ct)C_n} + \frac{M_z^R}{C_r - ct} \tag{4.1}$$

The analytic solutions

$$r = r_0 - \frac{M_r}{C_n}t, \quad \sigma = \sigma_0 + s_1 t + s_2 \ln(1 - c_1 t); \quad s_1 = \frac{M_r}{C_n}, \quad s_2 = -\frac{M_r + M_z^R}{c}, \quad c_1 = \frac{c}{C_r} \tag{4.2}$$

follow from the last two equations of (4.1) and

$$\varphi = \varphi_0 + r_0 t - \frac{M_r}{2C_n}t^2 \tag{4.3}$$

can be obtained from kinematic Eq. (1.9).

Using the second equation of (4.1), the solutions (4.2), relations (3.1) and (3.7) and, taking equality (1.10) into account, the derivative for the phase Φ can be represented in the form

$$\dot{\Phi} = \frac{1}{A - at - \dfrac{\nu^2 l_r^2 t^2}{m - \nu t}}\left[(C_n + C_r - ct - A + at)\left(\frac{M_r}{C_n}t - r_0\right) - \right.$$

$$\left. - (C_r - ct)(\sigma_0 + s_1 t + s_2 \ln(1 - c_1 t))\right] + \frac{M_r}{C_n}t - r_0; \quad m = m_n + m_r(0) \tag{4.4}$$

By differentiating equality (4.4), it is possible to obtain the function $P(t)$ in explicit form, which enables us to carry out the subsequent investigation of the nutational-precessional motion of the gyrostat in the whole space. However, the form of the function $P(t)$ obtained is unwieldy and its use for analysis and, in particular, for the synthesis of the modes of motion turns out to be extremely laborious.

We will now consider the problem of synthesizing the parameters of a gyrostat and the initial conditions that ensure motion for which the first evolution of the phase trajectory will take place with an increase in its curvature, which corresponds to the existence of an initial stage of motion with decreasing amplitude of the angle of nutation. We shall also assume that there is no initial rotation of the body-carrier about the longitudinal axis ($r_0 = 0$).

In order to solve the problem, it is useful to replace the function $P(t)$ with an approximating polynomial by expanding $P(t)$ in a Maclaurin series. Taking just the first two terms of the series, we obtain

$$P(t) = \dot{\Phi}\ddot{\Phi} \cong f_0 + f_1 t$$

$$f_0 = \frac{C_r}{A^3}[(aC_r - cA)\sigma_0^2 + AM_z^R\sigma_0]$$

$$f_1 = \frac{3a^2C_r^2\sigma_0^2}{A^4} + \frac{C_r\sigma_0}{A^3}\left[4aM_z^R + 2\sigma_0\left(\frac{C_r\nu^2 l_r^2}{m} - 2ca\right)\right] + \frac{1}{A^2}[c^2\sigma_0^2 - \sigma_0 c(2M_r + 3M_z^R) + (M_z^R)^2] \tag{4.5}$$

It was pointed out in Section 2 that stability of the polynomial $P_N(t)$, which, in the case being considered, reduces to the positiveness of the coefficients f_0 and f_1, is a sufficient condition for motion with a monotonic character of the PT.

Suppose $\sigma_0 > 0$, $M_z^R > 0$. Positiveness of f_0 is attained when inequality (3.6) is satisfied or the following inequalities

$$c/C_r > a/A, \quad \sigma_0(cA - aC_r) < AM_z^R \tag{4.6}$$

are jointly satisfied. The positiveness of f_1 is possible when the following conditions are jointly satisfied

$$\sigma_0\left(c - \frac{C_r\nu^2 l_r^2}{2am}\right) < M_z^R, \quad 3M_z^R - \frac{(M_z^R)^2}{c\sigma_0} < c\sigma_0 - M_r \tag{4.7}$$

and it is also possible when

$$C_r\nu^2 l_r^2 > 2cam, \quad M_r + 3M_z^R < 0$$

Other constraints can also be found which ensure the positiveness of f_0 and f_1. Similar conditions can also be obtained for other combinations of the signs of σ_0 and M_z^R in an analogous manner.

In order to illustrate conditions (4.6) and (4.7), we present the results of calculations of PTs in four cases (Fig. 3) obtained by numerical integration of Eqs. (4.1) and (1.9) for the following values of the parameters

$$A_n = A_r = 2.5\ \text{kg}\cdot\text{m}^2,\ C_n = 1\ \text{kg}\cdot\text{m}^2,\ C_r = 1.5\ \text{kg}\cdot\text{m}^2,\ a = c = 0.08\ \text{kg}\cdot\text{m}^2/\text{s}$$

$$l_r = 0.5\ \text{m},\ m_n = m_r = 35\ \text{kg},\ \nu = 1\ \text{kg/s},\ T = 20\ \text{s}$$

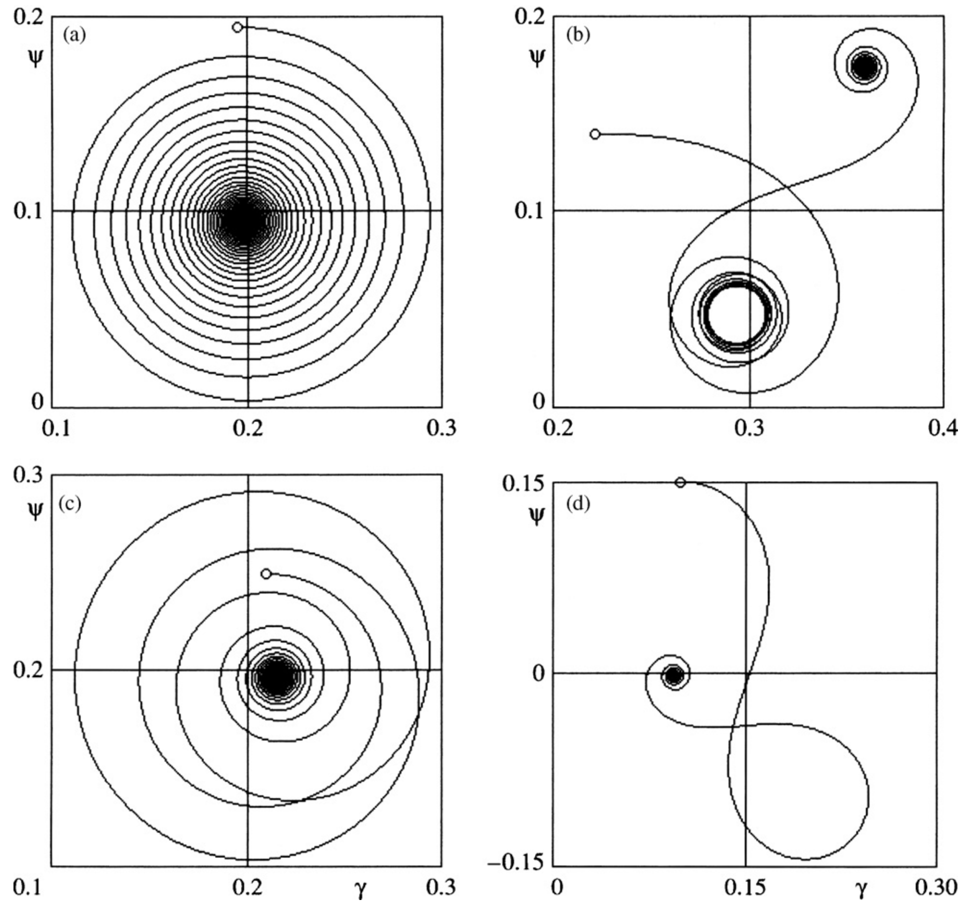

Fig. 3.

The initial conditions of the motion are shown below

Cases	*a*	*b*	*c*	*d*
M_r, N · m	1	–10	200	650
M_z^R, N · m	2	10	0.35	0.2
σ_0, 1/s	10	1	20	9.5
G_0, 1/s	0.3	0.15	0.35	0.18

Cases *a* and *b* correspond to all the conditions (4.6) and (4.7) being simultaneously satisfied. Here, in the initial stage, the trajectories have the form of a curling spiral and, for case *a*, this evolution is unique. In cases *c* and *d*, the second ones of the paired inequalities (4.6) and (4.7) are not satisfied and there are initial stages of motion along unwinding spirals.

5. The motion of a gyrostat of variable structure when acted upon by external moments of dissipative forces

We will consider the motion of an unbalanced gyrostat of variable structure acted upon by the constant longitudinal moments of internal interaction forces, the moments of reactive forces and, also, when there are external moments of dissipative forces, proportional to the angular velocities of the gyrostat. The motion is considered in a finite time interval $t \in [0, T]$ in which all of the inertia-mass parameters of the system, which vary with time, remain strictly positive (see Section 1). This formulation generalizes similar problems of the investigation of the motion of a rigid body and a gyrostat of constant mass, considered earlier (Refs. 6–9, etc).

In the above-mentioned case, the moments of the external dissipative forces have the form

$$M_x^e = -\nu p, \quad M_y^e = -\nu q, \quad M_{z,n}^e = -\lambda r, \quad M_{z,r}^e = -\mu(r+\sigma)$$

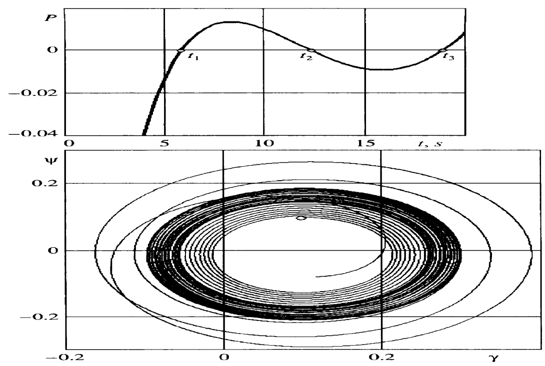

Fig. 4.

The constants υ, μ and λ characterize the action of the resistive medium on the gyrostat. The dynamic equations of motion (1.1) are written as follows:

$$\bar{A}(t)\dot{p} + (C(t) - A(t))qr + C_r(t)q\sigma = -\nu p$$
$$\bar{A}(t)\dot{q} - (C(t) - A(t))pr - C_r(t)p\sigma = -\nu q$$
$$C(t)\dot{r} + C_r(t)\dot{\sigma} = M_z^R - \lambda r - \mu(r+\sigma), \quad C_r(t)(\dot{r} + \dot{\sigma}) = M_r + M_z^R - \mu(r+\sigma) \tag{5.1}$$

where $\bar{A}(t) = A(t) - m\rho^2(t) > 0$ in the whole time interval $t \in [0, T]$ of the motion considered, since it is assumed that all the conditions for the correctness of the mechanical formulation of the problem are satisfied.

The equation and general solution for the absolute longitudinal velocity of the rotor

$$\Omega = r + \sigma, \quad \dot{\Omega} = \frac{1}{C_r(t)}(M_r + M_z^R - \mu\Omega)$$
$$\Omega(t) = \frac{1}{\mu}(M_r + M_z^R) - \frac{1}{\mu}(M_r + M_z^R - \mu\Omega_0)\exp[-\mu J_C(t)], \quad J_C(t) = \int_0^t \frac{dt}{C_r(t)} \tag{5.2}$$

follow from the last equation of (5.1).

Taking account of relations (5.2), the equation and general solution for the longitudinal velocity of the body-carrier

$$C_n\dot{r} = -M_r - \lambda r, \quad r(t) = \left(r_0 + \frac{M_r}{\lambda}\right)\exp\left[\frac{-\lambda t}{C_n}\right] - \frac{M_r}{\lambda} \tag{5.3}$$

follow from the third equation of (5.1).

In the case considered, the perturbing functions in Eq. (1.6) have the form

$$f_G(G, F) = -\nu G, \quad f_F(G, F) = 0$$

and the first two equations of (1.6) are therefore written as follows:

$$\dot{G} = -\frac{\nu G}{\bar{A}(t)}, \quad \dot{F} = -\frac{1}{\bar{A}(t)}(C_n r + C_r(t)\Omega - A(t)r) \tag{5.4}$$

The solution for the amplitude of the transverse angular velocity

$$G(t) = G_0\exp[-\nu J_A(t)], \quad G_0 > 0, \quad J_A(t) = \int_0^t \frac{1}{\bar{A}(t)} \tag{5.5}$$

follows from this.

Suppose that all the inertia-mass parameters are described by polynomial functions of time, which take strictly positive values:

$$A(t) = \sum_{i=0}^{k} a_i t^i, \quad C_r(t) = \sum_{i=0}^{m} c_i t^i, \quad m\rho^2(t) = \sum_{i=0}^{l} s_i t^i, \quad \bar{A}(t) = \sum_{i=0}^{n} (a_i - s_i)t^i = \sum_{i=0}^{n} \bar{a}_i t^i$$
$$n = \max\{k, l\} \tag{5.6}$$

in the interval $t \in [0, T]$.

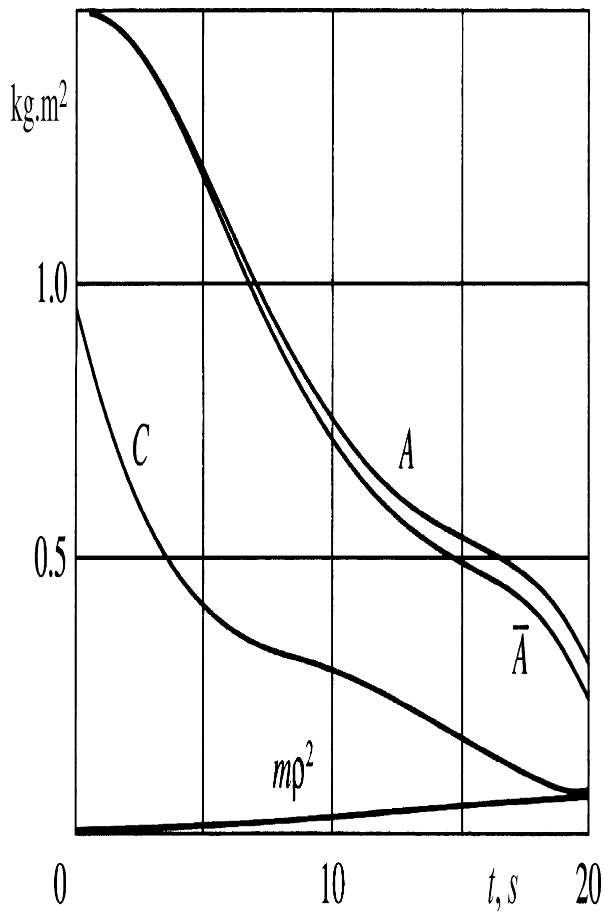

Fig. 5.

It is then possible to evaluate the integrals in relations (5.5) and (5.2) analytically. We obtain

$$J_A(t) = \sum_{i=1}^{n} \frac{\ln|t-\alpha_i|}{\dot{\bar{A}}(\alpha_i)}\Bigg|_0^t, \quad J_C(t) = \int_0^t \frac{dt}{C_r(t)} = \sum_{i=1}^{m} \frac{\ln|t-\beta_i|}{\dot{C}_r(\beta_i)}\Bigg|_0^t \tag{5.7}$$

where α_i and β_i are the roots of the polynomials $\bar{A}(t)$ and $C_r(t)$, among which, by virtue of the requirements imposed concerning the correctness of the mechanical formulation of the problem, there are no real roots falling in the interval $t \in [0, T]$. Note that formulae (5.7) can be verified by differentiation.

Taking account of formula (5.7), we obtain the final analytical solutions for the longitudinal angular velocities of the bodies and the amplitude of the transverse angular velocity of the gyrostat from expressions (5.2), (5.3) and (5.5). Using the first equation of (5.4), we have

$$P(t) = G\left(\frac{1}{2}\frac{d\dot{\Phi}^2}{dt} + \frac{\nu\dot{\Phi}^2}{\bar{A}(t)}\right) \tag{5.8}$$

We will first consider the case when there are no internal interaction and reactive force moments and the body-carrier does not have an initial longitudinal angular velocity[4]:

$$M_r = M_z^R = 0; \quad r_0 = 0 \tag{5.9}$$

We carry out subsidiary transformations of the quantities occurring in formula (5.8). When account is taken of inequalities (5.9), it follows from solution (5.3) that $r(t) \equiv 0$. Therefore, on the basis of relations (1.10) and (1.9), $\dot{\Phi} = \dot{F}$. Replacing $\dot{\Omega}$ by the right-hand side of the corresponding equation (5.2) and $\dot{F}$ by the right-hand side of (5.4), we write the expressions

$$\dot{\Phi}^2 = \frac{C_r^2(t)}{\bar{A}^2(t)}\Omega^2, \quad \frac{d\dot{\Phi}^2}{dt} = \frac{2\Omega^2}{\bar{A}^3}(\dot{C}_r C_r \bar{A} - \dot{\bar{A}} C_r^2 - \mu C_r \bar{A})$$

Table 1

i	0	1	2	3	4	5	6
$a_i \cdot 10^3$, kg·m^2/s^i	$1.5 \cdot 10^3$	14	−22.9	1.84	−0.044	0	0
$\bar{a}_i \cdot 10^3$, kg·m^2/s^i	$1.5 \cdot 10^3$	14	−23.3	1.85	−0.044	0	0
$c_i \cdot 10^3$, kg·m^2/s^i	$0.96 \cdot 10^3$	−198	23.9	−1.36	0.028	0	0
$s_i \cdot 10^3$, kg·m^2/s^i	0	0.3	0.4	−0.01	0	0	0
$p_i \cdot 10^3$, kg^2·m^4/s^{i+1}	$-0.29 \cdot 10^3$	107	−14.1	0.93	−0.04	$-0.8 \cdot 10^{-3}$	$-0.9 \cdot 10^{-5}$

The function (5.8) now takes the form

$$P(t) = \frac{G\Omega^2 C_r}{\bar{A}^3}(\dot{C}_r\bar{A} - \dot{\bar{A}}C_r - \mu\bar{A} + \nu C_r) \tag{5.10}$$

It follows from the solutions (5.5) and (5.2) that $G > 0$, $\Omega \neq 0$, and the factor in front of the brackets in expression (5.10) is therefore strictly positive over the whole interval $t \in [0, T]$. On the basis of the last remark, the expression

$$P(t) = \dot{C}_r\bar{A} - \dot{\bar{A}}C_r - \mu\bar{A} + \nu C_r \tag{5.11}$$

can be taken for the function $P(t)$.

It follows from relations (5.11) and (5.6) that the function $P(t)$ is a polynomial of degree $N = m + n - 1$, and the number of evolutions of the PT therefore cannot be greater than $N + 1$, if all the N roots of this polynomial prove to be positive and fall inside the interval $t \in [0, T]$.

The results of the numerical calculation of the PT and the polynomial $P(t)$ (the point corresponds to the start of the PT) are shown in Fig. 4 for the hypothetical laws of variation of the inertia-mass parameters (Fig. 5) corresponding to the fourth-order polynomials (5.6) with the coefficients presented in Table 1. Since, in the case considered, the leading coefficient of the polynomial $P(t)$ is equal to zero $p_7 = 4c_4\bar{a}_4 - 4\bar{a}_4c_4 = 0$, the polynomial is of the sixth order (the coefficients p_i are also given in Table 1). The calculations were carried out with conditions (5.9) being satisfied and the following values of the system parameters and the initial conditions of the motion

$$G_0 = 3.2, \quad \Omega_0 = 45\ 1/\text{s}, \quad \nu = 0.05, \quad \mu = 0.017, \quad \lambda = 0.3\ \text{kg}\cdot\text{m}^2/\text{s}, \quad T = 20\ \text{s}$$

The polynomial $P(t)$ has four real and two complex roots:

$$t_1 = 5.795, \quad t_2 = 12.32, \quad t_3 = 18.76, \quad t_4 = 31.13, \quad t_{5,6} = 11.96 \pm 24.99i$$

Since three real roots fall in the time interval considered $t \in [0, 20]$, we have an alternation of the four evolutions of the PT: when $t \in (0, t_1)$ and $t \in (t_2, t_3)$, they are unwinding spirals, and, when $t \in (t_1, t_2)$ and $t \in (t_3, T)$, they are curling spirals.

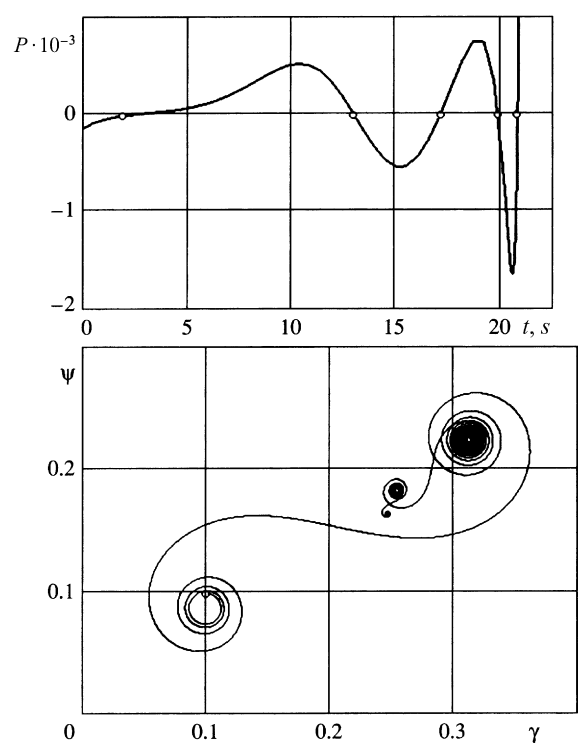

Fig. 6.

From the function (5.11), it is possible to obtain the following constraints, which are imposed on the laws of variation of the moments of inertia and ensure the curling of the PT and, consequently, a reduction in the amplitude of the angle of nutation

$$C(t)/\bar{A}(t) > \dot{C}(t)/\dot{\bar{A}}(t), \quad C(t)/\bar{A}(t) > \mu/\nu \tag{5.12}$$

In particular, condition (3.6) obtained above for a gyrostat of variable structure in the case of linear laws for the change in the moments of inertia when there are no dissipative moments ($\nu = \mu = 0$) follows from these conditions.

In concluding, we present the result of a further numerical calculation of the function $P(t)$ (5.8) and the PT (Fig. 6) in the case when conditions (5.9) are not satisfied. The calculations were carried out for polynomial dependences of the moments of inertia with the coefficients from Table 1 for the following values of the system parameters and the initial conditions of motion

$$r_0 = 10, \quad G_0 = 0.3, \quad \Omega_0 = 25\ 1/\mathrm{s}, \quad \nu = 0.02, \quad \mu = 0.01, \quad \lambda = 0.03\ \mathrm{kg\cdot m^2/s}$$

$$M_r = -10, \ M_z^R = -20\ \mathrm{kg\cdot m^2/s^2}, \quad T = 21.5\ \mathrm{s}$$

It can be seen that the function $P(t)$ (5.8) has five roots and that the phase trajectory has six evolutions.

Hence, a method for the investigating the behaviour of phase trajectories using the function $P(t)$ has been developed and employed for the qualitative analysis of processes in the dynamics of the motion of coaxial bodies and unbalanced gyrostats of variable structure. Some of the results may be useful in describing the motion about the centre of mass of spacecraft and satellite-gyrostats executing active manoeuvres.

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